msp

Algebraic & Geometric Topology 22 (2022) 1255–1272

Lower bounds for volumes and orthospectra of hyperbolic manifolds with geodesic boundary

Mikhail Belolipetsky
Martin Bridgeman

We derive explicit estimates for the functions which appear in the previous work of Bridgeman and Kahn. As a consequence, we obtain an explicit lower bound for the length of the shortest orthogeodesic in terms of the volume of a hyperbolic manifold with totally geodesic boundary. We also give an alternative derivation of a lower bound for the volumes of these manifolds as a function of the dimension.

32Q45

1 Introduction

Let M be a compact hyperbolic n–dimensional manifold with nonempty totally geodesic boundary. An *orthogeodesic* of M is a geodesic arc with endpoints in ∂M which are perpendicular to ∂M at the endpoints. The *orthospectrum* Λ_M of M is the set (with multiplicities) of lengths of orthogeodesics. As the orthogeodesics of M correspond to a subset of the closed geodesics of its double, the set of orthogeodesics of M is countable. We let $\mathrm{Vol}(M)$ and $\mathrm{Vol}(\partial M)$ be the volumes of the hyperbolic manifolds M and ∂M. We further let $\mathrm{L}(M)$ be the length of the shortest orthogeodesic of M. We will explore the relation between the three quantities $\mathrm{Vol}(M)$, $\mathrm{Vol}(\partial M)$ and $\mathrm{L}(M)$.

The orthospectrum was first introduced by Basmajian [2], who showed that a totally geodesic hypersurface S in a hyperbolic manifold can be decomposed into embedded disks which are in one-to-one correspondence with the orthogeodesics of the manifold M obtained by cutting along the hypersurface S. Then, by describing the radii of the disks in terms of the length of the corresponding orthogeodesics, Basmajian obtained the orthospectrum identity

$$\mathrm{Vol}(S) = \sum_{l\in\Lambda_M} V_{n-1}\big(\log\coth\big(\tfrac{1}{2}l\big)\big),$$

where $V_n(r)$ is the volume of a hyperbolic ball of radius r in $\mathbb{H}^n$.

Published: 25 August 2022 DOI: 10.2140/agt.2022.22.1255

Using a decomposition of the tangent bundle via orthogeodesics, the second author and Kahn proved the following:

Theorem 1 (Bridgeman and Kahn [6]) *Given $n \geq 2$, there exists a continuous monotonically decreasing function $F_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that, if M is a compact hyperbolic n–manifold with nonempty totally geodesic boundary, then*

$$\mathrm{Vol}(M) = \sum_{l \in \Lambda_M} F_n(l).$$

The function F_n is given by an integral formula; see (4) below. The above theorem was generalized to noncompact finite-volume hyperbolic manifolds with totally geodesic boundary by Vlamis and Yarmola [13].

An analysis of the asymptotic behavior of $F_n(l)$ as $l \to 0$ gives:

Theorem 2 (Bridgeman and Kahn [6]) *For $n \geq 3$, there exists a monotonically increasing function $H_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $C_n > 0$ such that, if M is a compact hyperbolic n–manifold with totally geodesic boundary with $\mathrm{Vol}(\partial M) = A$, then*

$$\mathrm{Vol}(M) \geq H_n(A) \geq C_n \cdot A^{(n-2)/(n-1)}.$$

The functions F_n and H_n and the implied constants C_n which appear in [6] are defined by complicated formulas and it is difficult to evaluate or estimate them. We resolve this issue and find explicit lower bounds in terms of the dimension n. We first prove the following relation between $\mathrm{Vol}(M)$ and $\mathrm{L}(M)$:

Theorem 3 *For $n \geq 3$, if M is a compact hyperbolic n–manifold M with totally geodesic boundary, then either $\mathrm{L}(M) \geq \frac{1}{2}\log\frac{5}{2}$ or*

$$e^{\mathrm{L}(M)} - 1 \geq g_n \sqrt{\frac{2\pi e}{n-1}} \cdot (\mathrm{Vol}(M))^{-1/(n-2)}, \tag{1}$$

where g_n is an explicit monotonically increasing function tending to 1.

The function g_n is given by (6) below. In particular, the first few approximate values are $g_3 = 0.120822$, $g_4 = 0.464543$, $g_5 = 0.563796$, $g_6 = 0.617183$.

One consequence of Theorem 3 is the following dichotomy between volume and shortest orthogeodesic:

Corollary 4 *Let M be a compact hyperbolic manifold with nonempty totally geodesic boundary of dimension $n \geq 3$. Then either*

$$\mathrm{Vol}(M) \geq 1 \quad \text{or} \quad e^{\mathrm{L}(M)} - 1 \geq \min\left(\sqrt{5/2} - 1, g_n \sqrt{\frac{2\pi e}{n-1}}\right).$$

The results of [6] have a number of applications that can be made more precise now. For example, Belolipetsky and Thomson [5] used them to estimate the volumes of hyperbolic manifolds with small systole constructed there. Inequality (1) allows us to restate the inequality from [5, Theorem 1.2]:

Corollary 5 *Hyperbolic manifolds with small systole constructed by Belolipetsky–Thomson in* [5] *satisfy*

$$\mathrm{Vol}(M) \geq \left(\tfrac{1}{2} g_n \sqrt{\frac{2\pi e}{n-1}} \cdot \frac{1}{\mathrm{Syst}_1(M)}\right)^{n-2}.$$

We also use our analysis to investigate the relation between $\mathrm{Vol}(M)$ and $\mathrm{Vol}(\partial M)$, which we compare with the results of Miyamoto in [12]. We prove:

Theorem 6 *Let M be a compact hyperbolic manifold with nonempty totally geodesic boundary of dimension $n \geq 3$. Then either*

$$\mathrm{Vol}(M) \geq \tfrac{1}{4} \log \tfrac{5}{2} \mathrm{Vol}(\partial M) \quad \text{or} \quad \mathrm{Vol}(M) \geq \tfrac{1}{3} h_n \sqrt{\frac{2\pi e}{n-1}} (\mathrm{Vol}(\partial M))^{(n-2)/(n-1)}, \tag{2}$$

where h_n is an explicit monotonically increasing function tending to 1.

The function h_n is given by (7), with the first few approximate values $h_3 = 0.203335$, $h_4 = 0.448875$, $h_5 = 0.542675$, $h_6 = 0.601147$.

In earlier work Miyamoto obtained a lower bound for the volume in terms of a linear function of the volume of the boundary:

Theorem 7 (Miyamoto [12, Theorem 4.2]) *Let M be a hyperbolic n–manifold with totally geodesic boundary. Then there are constants $\rho_n > 0$ such that*

$$\mathrm{Vol}(M) \geq \rho_n \cdot \mathrm{Vol}(\partial M). \tag{3}$$

One application of both (2) and (3) is to obtain lower bounds on the volume of a hyperbolic manifold with totally geodesic boundary in terms of the dimension. Although both use very different methods, their resulting bounds are surprisingly similar.

For n even, applying the Gauss–Bonnet formula for the double DM gives

$$\mathrm{Vol}(M) = \tfrac{1}{2}\,\mathrm{Vol}(DM) = \tfrac{1}{4}|\chi(DM)|V_n \geq \tfrac{1}{4}V_n,$$

where V_n is the volume of the unit n–sphere in $\mathbb{R}^{n+1}$. For n odd, both (2) and (3) can be used to leverage the Gauss–Bonnet theorem on the boundary to give lower bounds for the volume of the manifolds.

In [9], Kellerhals used packing estimates to show that Miyamoto's function ρ_n is monotonically increasing with the approximate values $\rho_3 = 0.29156$, $\rho_4 = 0.43219$, $\rho_5 = 0.54167$, $\rho_6 = 0.64652$.

Thus, for M a hyperbolic n–manifold with nonempty totally geodesic boundary and n odd, we have

$$\mathrm{Vol}(M) \geq \tfrac{1}{2}\rho_n V_{n-1}.$$

Using our bound in (2), we can derive a similar estimate. We prove:

Theorem 8 *Let M be a hyperbolic n–manifold with nonempty totally geodesic boundary and n odd. Then*

$$\mathrm{Vol}(M) \geq \min\big(\tfrac{1}{8}\log\tfrac{5}{2}, \tfrac{1}{6}h_n\big)V_{n-1}.$$

The paper is organized as follows. We first describe the functions $F_n(x)$ and $M_n(x)$ and, by a careful analysis, obtain uniform lower bounds for each as functions of n and x. An important step is bounding an incomplete Beta function which requires us to restrict to $x \leq \frac{1}{2}\log\frac{5}{2}$ (see Lemma 12). We then apply these bounds to prove the bounds on volume and ortholength in Theorems 3 and 6 above. In Section 5 we consider more carefully the 3–dimensional case. In Section 6 we conclude with the proof of Theorem 8 and a related discussion.

Acknowledgments We thank Ruth Kellerhals for helpful correspondence. We would also like to thank the referee for their comments and insights which improved the paper.

Belolipetsky is partially supported by CNPq, FAPERJ and MPIM in Bonn. Bridgeman is supported by NSF grant DMS-2005498 and by a grant from the Simons Foundation (675497, MJB).

2 The functions F_n and M_n

In previous work, an integral formula for F_n is derived: We let V_k be the volume of the unit k–sphere in $\mathbb{R}^{k+1}$. Then, from [6], we have[1]

$$F_n(l) = \frac{2^{n-1} V_{n-2} V_{n-3}}{V_{n-1}} \int_0^1 \frac{r^{n-3}}{(\sqrt{1-r^2})^{n-2}} M_n\left(\sqrt{\frac{e^{2l}-r^2}{1-r^2}}\right) dr, \tag{4}$$

where

$$M_n(b) = \int_{-1}^1 du \int_b^\infty \frac{\log((v^2-1)(u^2-b^2)/(v^2-b^2)(u^2-1))}{(v-u)^n}\, dv. \tag{5}$$

Furthermore, it is shown that the function $M_n(b)$ can be given in terms of standard functions. In order to describe this function, we define the following: For $n \geq 1$ we define the polynomial function P_n by

$$P_n(x) = \sum_{k=1}^n \frac{x^k}{k}.$$

We also define $P_0(x) = 0$. We note that for $|x| < 1$, $P_n(x)$ is the first n terms of the Taylor series of $-\log(1-x)$. We therefore define the function $L_n(x)$ by $L_n(x) = \log|1-x| + P_n(x)$. For $|x| < 1$ we have

$$L_n(x) = -\sum_{k=n+1}^\infty \frac{x^k}{k}.$$

We note that $L_0(x) = \log|1-x|$. We also note that $P_n(1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, the n[th] Harmonic number. Using these functions, M_n can be written down explicitly:

Lemma 9 (Bridgeman and Kahn [6, Lemma 7]) *The function $M_n\colon (1,\infty) \to \mathbb{R}_+$ has the explicit form*

$$\begin{aligned}
&(n-1)(n-2)M_n(b)\\
&= \frac{1}{(b-1)^{n-2}}\left(\log\frac{(b+1)^2}{4b} + 2P_{n-2}(1) - L_{n-3}\left(\frac{b-1}{b+1}\right) - (-1)^n L_{n-3}\left(\frac{-b+1}{b+1}\right)\right)\\
&\quad + \frac{1}{(b+1)^{n-2}}\left(-\log\frac{(b-1)^2}{4b} - 2P_{n-2}(1) + L_{n-3}\left(\frac{b+1}{b-1}\right) + (-1)^n L_{n-3}\left(\frac{-b-1}{b-1}\right)\right)\\
&\quad + \frac{1}{(2b)^{n-2}}\left(L_{n-3}\left(\frac{2b}{b+1}\right) - L_{n-3}\left(\frac{2b}{b-1}\right)\right)\\
&\quad + \frac{1}{2^{n-2}}\left(L_{n-3}\left(\frac{2}{b+1}\right) - (-1)^n L_{n-3}\left(\frac{-2}{b-1}\right)\right).
\end{aligned}$$

[1] The original formula had an incorrect factor of 2 rather than 2^{n-1}, which was corrected by Theorem 2.1 of [13].

Furthermore, M_n *satisfies*

$$\lim_{b\to 1^+}(b-1)^{n-2}M_n(b)=\frac{2P_{n-2}(1)}{(n-1)(n-2)} \quad \textit{and} \quad \lim_{b\to\infty}\frac{b^{n-1}}{\log b}M_n(b)=\frac{4}{n-1}.$$

We note that the above is a consequence of the following formulas:

Lemma 10 (Bridgeman and Kahn [6, Corollary 6]) *For* $n\geq 2$

$$\int\frac{\log|x-a|}{(x-b)^n}\,dx=\frac{1}{n-1}\left(\frac{L_{n-2}((a-b)/(x-b))}{(a-b)^{n-1}}-\frac{\log|x-a|}{(x-b)^{n-1}}\right).$$

Furthermore, *for* $k\geq 1$,

$$\lim_{x\to a}\left(\frac{\log|x-a|}{(b-x)^k}-\frac{L_n((b-a)/(b-x))}{(b-a)^k}\right)=\frac{\log|b-a|-P_n(1)}{(b-a)^k}.$$

3 Explicit lower bounds for F_n and M_n

In this section, we give explicit lower bounds on for the functions F_n and M_n. As these functions are only defined for $n\geq 3$, in the following a standing assumption is that $n\geq 3$. In order to obtain our bounds, we need to derive a lower bound on $M_n(b)$ which is uniform both in n and b. By Lemma 9, we have

$$\lim_{b\to 1^+}(b-1)^{n-2}M_n(b)=\frac{2P_{n-2}(1)}{(n-1)(n-2)}.$$

We prove the following uniform lower bound:

Lemma 11 *For* $b\in(1,2]$,

$$(b-1)^{n-2}M_n(b)\geq\frac{P_{n-3}(1)+\left(1-1/3^{n-2}\right)\left(P_{n-2}(1)+\log\frac{3}{4}\right)}{(n-1)(n-2)}.$$

Proof From (5), for M_n we have that

$$\begin{aligned}M_n(b)&=\int_{-1}^{1}du\int_b^\infty\frac{\log((v^2-1)(b^2-u^2)/(v^2-b^2)(1-u^2))}{(v-u)^n}\,dv\\&\geq\int_{-1}^{1}du\int_b^\infty\frac{\log((v^2-1)(b-u)/(v^2-b^2)(1-u))}{(v-u)^n}\,dv\end{aligned}$$

as $b+u>1+u$. We split the interior integral on the right into two integrals,

$$I_1=-\int_b^\infty\frac{\log(v-b)}{(v-u)^n}\,dv,\quad I_2=\int_b^\infty\frac{\log((v^2-1)(b-u)/(v+b)(1-u))}{(v-u)^n}\,dv.$$

By Lemma 10, we have

$$\begin{aligned} I_1 &= \frac{1}{n-1}\left(\frac{\log(v-b)}{(v-u)^{n-1}} - \frac{L_{n-2}((b-u)/(v-u))}{(b-u)^{n-1}}\right)\Bigg|_b^\infty \\ &= \frac{1}{n-1}\lim_{v\to b+}\left(\frac{L_{n-2}((b-u)/(v-u))}{(b-u)^{n-1}} - \frac{\log(v-b)}{(v-u)^{n-1}}\right). \end{aligned}$$

By the limit in Lemma 10, we have

$$I_1 = \frac{1}{n-1}\left(\frac{P_{n-2}(1)-\log(b-u)}{(b-u)^{n-1}}\right).$$

Integrating by parts, we get

$$\begin{aligned} I_2 = -\frac{1}{n-1}\Bigg(&\frac{\log((v^2-1)(b-u)/(v+b)(1-u))}{(v-u)^{n-1}}\Bigg|_b^\infty \\ &+\int_b^\infty \frac{dv}{(v-u)^{n-1}}\left(\frac{1}{v-1}+\frac{1}{v+1}-\frac{1}{v+b}\right)\Bigg). \end{aligned}$$

As $v+b > v+1$, we have

$$\frac{1}{v-1}+\frac{1}{v+1}-\frac{1}{v+b} \geq \frac{1}{v-1} > 0.$$

Therefore,

$$\begin{aligned} I_2 &\geq -\frac{1}{n-1}\left(\frac{\log((v^2-1)(b-u)/(v+b)(1-u))}{(v-u)^{n-1}}\Bigg|_b^\infty\right) \\ &= \frac{1}{n-1}\left(\frac{\log((b^2-1)(b-u)/(2b)(1-u))}{(b-u)^{n-1}}\right). \end{aligned}$$

Therefore, combining, we have

$$M_n(b) \geq \frac{1}{n-1}\left(\int_{-1}^1 \frac{\log((b^2-1)/2b(1-u)) + P_{n-2}(1)}{(b-u)^{n-1}}\,du\right) = J_1(b)+J_2(b),$$

where

$$\begin{aligned} J_1(b) &= \frac{1}{n-1}\left(\int_{-1}^1 \frac{\log((b^2-1)/2b) + P_{n-2}(1)}{(b-u)^{n-1}}\,du\right), \\ J_2(b) &= \frac{1}{n-1}\left(\int_{-1}^1 \frac{-\log(1-u)}{(b-u)^{n-1}}\,du\right). \end{aligned}$$

By integration, we have

$$J_1(b) = \frac{\log((b^2-1)/2b) + P_{n-2}(1)}{(n-1)(n-2)}\left(\frac{1}{(b-1)^{n-2}} - \frac{1}{(b+1)^{n-2}}\right).$$

Using Lemma 10, we get

$$J_2(b) = \frac{1}{(n-1)(n-2)}\left(\frac{-\log(1-u)}{(b-u)^{n-2}} + \frac{L_{n-3}((b-1)/(b-u))}{(b-1)^{n-2}}\right)\Bigg|_{-1}^{1}.$$

Therefore,

$$\begin{aligned}(n-1)(n-2)J_2(b) = {}&\frac{-L_{n-3}((b-1)/(b+1))}{(b-1)^{n-2}}\\ &+\frac{\log 2}{(b+1)^{n-2}}+\lim_{u\to 1-}\left(-\frac{\log(1-u)}{(b-u)^{n-2}}+\frac{L_{n-3}((b-1)/(b-u))}{(b-1)^{n-2}}\right).\end{aligned}$$

By Lemma 10, we have the limit

$$\lim_{u\to 1-}\left(-\frac{\log(1-u)}{(b-u)^{n-2}} + \frac{L_{n-3}((b-1)/(b-u))}{(b-1)^{n-2}}\right) = \frac{P_{n-3}(1)-\log(b-1)}{(b-1)^{n-2}}.$$

Combining, we get

$$(n-1)(n-2)J_2(b) = \frac{-\log(b-1)+P_{n-3}(1)-L_{n-3}((b-1)/(b+1))}{(b-1)^{n-2}} + \frac{\log 2}{(b+1)^{n-2}}.$$

Thus,

$$\begin{aligned}(n-1)(n-2)M_n(b) \geq {}&\frac{\log((b+1)/2b)+P_{n-2}(1)+P_{n-3}(1)-L_{n-3}((b-1)/(b+1))}{(b-1)^{n-2}}\\ &+\frac{\log(4b/(b^2-1))-P_{n-2}(1)}{(b+1)^{n-2}}.\end{aligned}$$

For $b \in (1,2]$, we have

$$\log\frac{b+1}{2b} + P_{n-2}(1) \geq \log\tfrac{3}{4}+1 > 0 \quad\text{and}\quad -L_{n-3}\left(\frac{b-1}{b+1}\right) > 0,$$

giving

$$\begin{aligned}(n-1)(n-2)M_n(b) \geq {}&\frac{P_{n-3}(1)}{(b-1)^{n-2}} + \frac{\log((b+1)/2b)+P_{n-2}(1)}{(b-1)^{n-2}}\\ &+\frac{\log(4b/(b^2-1))-P_{n-2}(1)}{(b+1)^{n-2}}.\end{aligned}$$

As $(b+1)/(b-1) \geq 3$ on $(1,2]$, we have

$$\begin{aligned}&\frac{\log((b+1)/2b)+P_{n-2}(1)}{(b-1)^{n-2}}\\ &= \left(1-\frac{1}{3^{n-2}}\right)\left(\frac{\log((b+1)/2b)+P_{n-2}(1)}{(b-1)^{n-2}}\right) + \frac{1}{3^{n-2}}\left(\frac{\log((b+1)/2b)+P_{n-2}(1)}{(b-1)^{n-2}}\right)\\ &\geq \left(1-\frac{1}{3^{n-2}}\right)\left(\frac{\log((b+1)/2b)+P_{n-2}(1)}{(b-1)^{n-2}}\right) + \frac{\log((b+1)/2b)+P_{n-2}(1)}{(b+1)^{n-2}}.\end{aligned}$$

Therefore,

$$(n-1)(n-2)M_n(b) \geq \frac{P_{n-3}(1)+(1-1/3^{n-2})\big(\log((b+1)/2b)+P_{n-2}(1)\big)}{(b-1)^{n-2}} + \frac{\log((b+1)/2b)+P_{n-2}(1)+\log(4b/(b^2-1))-P_{n-2}(1)}{(b+1)^{n-2}}.$$

This gives

$$(n-1)(n-2)M_n(b) \geq \frac{P_{n-3}(1)+(1-1/3^{n-2})\big(\log((b+1)/2b)+P_{n-2}(1)\big)}{(b-1)^{n-2}} + \frac{\log(2/(b-1))}{(b+1)^{n-2}}.$$

Finally,

$$(n-1)(n-2)M_n(b) \geq \frac{P_{n-3}(1)+(1-1/3^{n-2})\big(\log\tfrac{3}{4}+P_{n-2}(1)\big)}{(b-1)^{n-2}} \geq \frac{0.474879}{(b-1)^{n-2}}. \quad \square$$

With this bound in hand, we now find a lower bound for $F_n(x)$ by integration.

Lemma 12 *For $l \leq \frac{1}{2}\log\frac{5}{2}$, we have*

$$F_n(l) \geq \frac{K_n}{(e^l-1)^{n-2}},$$

where

$$K_n = \frac{\big(P_{n-3}(1)+(1-1/3^{n-2})\big(P_{n-2}(1)+\log\frac{3}{4}\big)\big)2^{n-2}V_{n-2}V_{n-3}\Gamma(\frac{1}{2}n)^2}{(n-2)^2V_{n-1}\Gamma(n)}.$$

Proof We let $a = e^l$. Then, by Lemma 11 above, we have

$$M_n\bigg(\sqrt{\frac{a^2-r^2}{1-r^2}}\bigg) \geq \frac{A_n}{\big(\sqrt{(a^2-r^2)/(1-r^2)}-1\big)^{n-2}} \quad \text{for } \sqrt{\frac{a^2-r^2}{1-r^2}} \leq 2,$$

where

$$A_n = \frac{P_{n-3}(1)+(1-1/3^{n-2})\big(P_{n-2}(1)+\log\frac{3}{4}\big)}{(n-1)(n-2)}.$$

Solving this for $r < \sqrt{\frac{1}{3}(4-a^2)} = r(a)$, we obtain

$$F_n(l) \geq \frac{2^{n-1}V_{n-2}V_{n-3}}{V_{n-1}} \int_0^{r(a)} \frac{r^{n-3}}{(\sqrt{1-r^2})^{n-2}} \frac{A_n}{\big(\sqrt{(a^2-r^2)/(1-r^2)}-1\big)^{n-2}}\,dr.$$

Simplifying, we get

$$F_n(l) \geq \frac{2^{n-1} A_n V_{n-2} V_{n-3}}{V_{n-1}} \int_0^{r(a)} \frac{r^{n-3}}{(\sqrt{a^2-r^2}-\sqrt{1-r^2})^{n-2}}\,dr$$
$$= \frac{2^{n-1} A_n V_{n-2} V_{n-3}}{V_{n-1}} \int_0^{r(a)} r^{n-3}\left(\frac{\sqrt{a^2-r^2}+\sqrt{1-r^2}}{a^2-1}\right)^{n-2} dr.$$

As $\sqrt{a^2-r^2}/\sqrt{1-r^2} \geq a$, we have $\sqrt{a^2-r^2}+\sqrt{1-r^2} \geq (a+1)\sqrt{1-r^2}$, giving

$$F_n(l) \geq \frac{2^{n-1} A_n V_{n-2} V_{n-3}}{(a-1)^{n-2} V_{n-1}} \int_0^{r(a)} r^{n-3}(\sqrt{1-r^2})^{n-2}\,dr.$$

Therefore,

$$F_n(l) \geq \frac{2^{n-1} A_n V_{n-2} V_{n-3}}{(a-1)^{n-2} V_{n-1}} \int_0^{r(a)} r^{n-3}(1-r^2)^{n/2-1}\,dr.$$

We change the variable to $t = r^2$ to get

$$F_n(l) \geq \frac{2^{n-2} A_n V_{n-2} V_{n-3}}{(a-1)^{n-2} V_{n-1}} \int_0^{r(a)^2} t^{n/2-2}(1-t)^{n/2-1}\,dt.$$

The Beta function $B(a,b)$ and the incomplete Beta function $B(x:a,b)$ are defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \quad B(x:a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt.$$

Therefore,

$$F_n(l) \geq \frac{2^{n-2} A_n V_{n-2} V_{n-3}}{(a-1)^{n-2} V_{n-1}} B\big(r(a)^2 : \tfrac{1}{2}n-1, \tfrac{1}{2}n\big).$$

We note that

$$B(a-1,a) = B\big(\tfrac{1}{2} : a-1, a\big) + B\big(\tfrac{1}{2} : a, a-1\big).$$

On $\big[0, \tfrac{1}{2}\big]$, as $t < 1-t$, we have $t^{a-1}(1-t)^{a-2} \leq t^{a-2}(1-t)^{a-1}$, giving

$$B\big(\tfrac{1}{2} : a-1, a\big) \geq B\big(\tfrac{1}{2} : a, a-1\big).$$

Thus, $B\big(\tfrac{1}{2} : a-1, a\big) \geq \tfrac{1}{2}B(a-1,a)$.

Therefore, if we let $r(a)^2 \geq \tfrac{1}{2}$, then

$$B\big(r(a)^2 : \tfrac{1}{2}n-1, \tfrac{1}{2}n\big) \geq \tfrac{1}{2}B\big(\tfrac{1}{2}n-1, \tfrac{1}{2}n\big) = \frac{\Gamma\big(\tfrac{1}{2}n-1\big)\Gamma\big(\tfrac{1}{2}n\big)}{2\Gamma(n-1)} = \frac{n-1}{n-2}\frac{\Gamma\big(\tfrac{1}{2}n\big)^2}{\Gamma(n)}.$$

For $r(a)^2 \geq \tfrac{1}{2}$, we require $a \leq \sqrt{5/2}$. Therefore, for $l \leq \tfrac{1}{2}\log\tfrac{5}{2}$, we have

$$F(l) \geq \frac{\big(P_{n-3}(1)+(1-1/3^{n-2})\big(P_{n-2}(1)+\log\tfrac{3}{4}\big)\big)2^{n-2} V_{n-2} V_{n-3} \Gamma\big(\tfrac{1}{2}n\big)^2}{(n-2)^2 V_{n-1}\Gamma(n)} \frac{1}{(e^l-1)^{n-2}}.$$

This concludes the proof. □

4 Systole and volume estimates

We now use the bound for $F(l)$ to obtain a lower bound on the length of the shortest orthogeodesic and to obtain lower bounds on volume in terms of the area of the boundary. We first will need the following elementary calculation:

Lemma 13 *The constants K_n from Lemma 12 satisfy*

$$K_n \geq \left(\frac{2\pi e}{n-1}\right)^{(n-1)/2} \frac{3\bigl(P_{n-3}(1) + (1-1/3^{n-2})\bigl(P_{n-2}(1) + \log \frac{3}{4}\bigr)\bigr)}{2^{3/2} e^{5/2} (n-2)}.$$

Proof The volumes of spheres are given by

$$V_n = \frac{(n+1)\pi^{(n+1)/2}}{\Gamma\bigl(\frac{1}{2}(n+3)\bigr)}.$$

We have Legendre's replacement formula

$$\Gamma(z)\Gamma\bigl(z+\tfrac{1}{2}\bigr) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Thus,

$$\begin{aligned}\frac{2^{n-2} V_{n-2} V_{n-3} \Gamma\bigl(\frac{1}{2}n\bigr)^2}{(n-2)^2 V_{n-1} \Gamma(n)} &= \frac{(n-1) 2^{n-2} \pi^{(n-3)/2} \Gamma\bigl(\frac{1}{2}(n+2)\bigr)\Gamma\bigl(\frac{1}{2}n\bigr)}{(n-2) n \Gamma\bigl(\frac{1}{2}(n+1)\bigr)\Gamma(n)} \\ &= \frac{(n-1)\pi^{(n-2)/2}\Gamma\bigl(\frac{1}{2}(n+2)\bigr)}{2(n-2)n\Gamma\bigl(\frac{1}{2}(n+1)\bigr)^2}.\end{aligned}$$

By using the upper and lower bounds for the Gamma function

$$\sqrt{2\pi}\, x^{x+1/2} e^{-x} \leq \Gamma(x+1) \leq e x^{x+1/2} e^{-x},$$

we obtain

$$\begin{aligned}\frac{(n-1)\pi^{(n-2)/2}\Gamma\bigl(\frac{1}{2}(n+2)\bigr)}{2(n-2)n\Gamma\bigl(\frac{1}{2}(n+1)\bigr)^2} &\geq \frac{(n-1)\pi^{(n-2)/2}\bigl(\sqrt{2\pi}\bigl(\frac{1}{2}n\bigr)^{(n+1)/2} e^{-n/2}\bigr)}{2(n-2)n\bigl(e^2\bigl(\frac{1}{2}(n-1)\bigr)^n e^{-(n-1)}\bigr)} \\ &= \frac{2^{n/2-1}\pi^{(n-1)/2} n^{(n-1)/2} e^{n/2-3}}{(n-2)(n-1)^{n-1}}.\end{aligned}$$

Thus,

$$K_n \geq \left(\frac{2\pi n e}{(n-1)^2}\right)^{(n-1)/2} \frac{P_{n-3}(1) + (1-1/3^{n-2})\bigl(P_{n-2}(1) + \log \frac{3}{4}\bigr)}{(n-2) e^{5/2}\sqrt{2}}.$$

Finally, $(n/(n-1))^{(n-1)/2}$ is monotonically increasing; then, as $n \geq 3$, we have $(n/(n-1))^{(n-1)/2} \geq \frac{3}{2}$, so

$$K_n \geq \left(\frac{2\pi e}{n-1}\right)^{(n-1)/2} \frac{3\big(P_{n-3}(1) + (1 - 1/3^{n-2})\big(P_{n-2}(1) + \log \frac{3}{4}\big)\big)}{2(n-2)e^{5/2}\sqrt{2}}. \qquad \square$$

We now can prove the bound in Theorem 3, which we restate below:

Theorem 3 *Let M be a compact hyperbolic n–manifold with totally geodesic boundary. Then either* $\mathrm{L}(M) > \frac{1}{2}\log\frac{5}{2}$ *or*

$$e^{\mathrm{L}(M)} - 1 \geq g_n \sqrt{\frac{2\pi e}{n-1}} (\mathrm{Vol}(M))^{-1/(n-2)},$$

where g_n is an explicit monotonically increasing function tending to 1.

Proof Let $L = \mathrm{L}(M)$. If $L \leq \frac{1}{2}\log\frac{5}{2}$, then, by Lemma 12,

$$\mathrm{Vol}(M) \geq F_n(L) \geq \frac{K_n}{(e^L - 1)^{n-2}}.$$

Solving the latter, we have

$$e^L - 1 \geq \left(\frac{K_n}{\mathrm{Vol}(M)}\right)^{1/(n-2)},$$

which gives

$$e^L - 1 \geq K_n^{1/(n-2)} \,\mathrm{Vol}(M)^{-1/(n-2)}.$$

Therefore, by Lemma 13, we have

$$e^L - 1 \geq g_n \sqrt{\frac{2\pi e}{n-1}} (\mathrm{Vol}(M))^{-1/(n-2)},$$

where

$$g_n = \left(\frac{3\sqrt{\pi}\big(P_{n-3}(1) + (1 - 1/3^{n-2})\big(P_{n-2}(1) + \log\frac{3}{4}\big)\big)}{2(n-2)(n-1)^{1/2}e^2}\right)^{1/(n-2)}. \tag{6} \qquad \square$$

We now obtain a lower bound on the volume in terms of the boundary area. We will need an auxiliary function S_n given by

$$S_n(x) = \int_0^x \cosh^{n-1}(r)\, dr.$$

We prove Theorem 6, which we first restate:

Theorem 6 *Let M be a hyperbolic manifold with totally geodesic boundary. Then either*

$$\mathrm{Vol}(M) \geq \tfrac{1}{4} \log \tfrac{5}{2} \, \mathrm{Vol}(\partial M)$$

or

$$\mathrm{Vol}(M) \geq \tfrac{1}{3} h_n \sqrt{\frac{2\pi e}{n-1}} \, \mathrm{Vol}(\partial M)^{(n-2)/(n-1)},$$

where h_n is an explicit monotonically increasing function tending to 1.

Proof Let $L = \mathrm{L}(M)$, $V = \mathrm{Vol}(M)$, $A = \mathrm{Vol}(\partial M)$. Then, by Theorem 1,

$$V \geq F_n(L).$$

Further, the totally geodesic boundary ∂M has embedded collar of radius $\frac{1}{2}L$. By elementary hyperbolic geometry, this embedded collar has volume $A \cdot S_n\big(\frac{1}{2}L\big)$. Thus,

$$V \geq A \cdot S_n\big(\tfrac{1}{2}L\big) \geq A \cdot \tfrac{1}{2}L.$$

It follows that

$$V \geq \max\big(F_n(L), A \cdot \tfrac{1}{2}L\big).$$

As $F_n(x)$ is monotonically decreasing and $\frac{1}{2}Ax$ monotonically increasing, we have a unique $l > 0$ satisfying

$$F_n(l) = A \cdot \tfrac{1}{2}l.$$

Furthermore, it follows that $V \geq A \cdot \frac{1}{2}l$. If $l \geq \frac{1}{2}\log\frac{5}{2}$, then

$$V \geq \tfrac{1}{4} \log \tfrac{5}{2} A,$$

giving the first inequality of the theorem.

Now assume that $l \leq \frac{1}{2}\log\frac{5}{2}$. Then, by Lemma 12,

$$V \geq \max\left(\frac{K_n}{(e^l - 1)^{n-2}}, A \cdot \tfrac{1}{2}l\right).$$

We therefore consider l_0, the unique solution of

$$\frac{K_n}{(e^{l_0} - 1)^{n-2}} = A \cdot \tfrac{1}{2}l_0.$$

We observe that $l_0 \leq l$ and therefore we have $l_0 \leq \frac{1}{2}\log\frac{5}{2}$. Solving

$$A \cdot \tfrac{1}{2}l_0 = \frac{K_n}{(e^{l_0} - 1)^{n-2}},$$

we obtain

$$(e^{l_0}-1)l_0^{1/(n-2)} = \left(\frac{2K_n}{A}\right)^{1/(n-2)}.$$

Thus, as $l_0 < \frac{1}{2}\log\frac{5}{2}$ and $(e^x-1)/x$ is monotonically increasing, we have $e^{l_0}-1 \le al_0$, where

$$a = \frac{\sqrt{5/2}-1}{\log\sqrt{5/2}} = 1.26846.$$

Hence we have

$$a\cdot l_0^{(n-1)/(n-2)} \ge \left(\frac{2K_n}{A}\right)^{1/(n-2)},$$

$$l_0 \ge \frac{1}{a^{(n-2)/(n-1)}}\left(\frac{2K_n}{A}\right)^{1/(n-1)} \ge \frac{1}{a}\left(\frac{K_n}{A}\right)^{1/(n-1)}.$$

Combining with the inequality for V, we get

$$V \ge A\cdot\tfrac{1}{2}l \ge A\cdot\tfrac{1}{2}l_0 \ge \frac{1}{2a}K_n^{1/(n-1)}A^{(n-2)/(n-1)}.$$

Hence, by Lemma 13 above,

$$V \ge \frac{h_n}{2a}\sqrt{\frac{2\pi e}{n-1}}A^{(n-2)/(n-1)},$$

where

$$h_n = \left(\frac{3\big(P_{n-3}(1)+(1-1/3^{n-2})\big(P_{n-2}(1)+\log\frac{3}{4}\big)\big)}{2^{3/2}e^{5/2}(n-2)}\right)^{1/(n-1)}. \tag{7}$$

For $n \ge 3$ it is easy to check that h_n is monotonically increasing to 1. Evaluating a, we get

$$V \ge \frac{h_n}{2.53692}\sqrt{\frac{2\pi e}{n-1}}A^{(n-2)/(n-1)} \ge \tfrac{1}{3}h_n\sqrt{\frac{2\pi e}{n-1}}A^{(n-2)/(n-1)}. \qquad \square$$

5 Dimension 3 case

We note that the constants in the main theorems proved for general dimension can be improved in any specific case by analyzing F_n individually. We now consider the 3–dimensional case separately.

In [11], Masai and McShane proved that the volume identity of Bridgeman and Kahn (see Theorem 1) is equal to the identity obtained by Calegari [7] using a different decomposition. Applying Calegari's formula in dimension 3, they obtained an elementary closed form for F_3, namely

$$F_3(x) = 2\pi\left(\frac{x+1}{e^{2x}-1}\right). \tag{8}$$

We note that there is a normalization error in [11] (by a factor of 4π) and the above formula is the corrected version (see [13], where the correct version is also stated).

Using the formula of Masai and McShane for F_3, we can give an elementary argument that improves the constants in Theorem 3 in the case of $n = 3$. We would like to thank the referee for this observation.

Propostion 14 *Let M be a compact hyperbolic 3–manifold with nonempty totally geodesic boundary. Then either $\mathrm{L}(M) > 1.25$ or*

$$e^{\mathrm{L}(M)} - 1 \geq \frac{\pi}{V(M)}.$$

Proof By elementary calculus for $0 \leq x \leq 1.25$, we have

$$\frac{x+1}{e^x+1} \geq \frac{1}{2}.$$

Thus, for $\mathrm{L}(M) \leq 1.25$, equation (8) gives

$$V(M) \geq F(\mathrm{L}(M)) = 2\pi\left(\frac{\mathrm{L}(M)+1}{e^{\mathrm{L}(M)}+1}\right)\frac{1}{e^{\mathrm{L}(M)}-1} \geq \frac{\pi}{e^{\mathrm{L}(M)}-1}.$$

Thus, if $\mathrm{L}(M) \leq 1.25$,

$$e^{\mathrm{L}(M)} - 1 \geq \frac{\pi}{V(M)}. \qquad \square$$

We now compare this with Theorem 3. For $n = 3$, the theorem states that if $\mathrm{L}(M) \leq \frac{1}{2}\log\frac{5}{2}$, then

$$e^{\mathrm{L}(M)} - 1 \geq \frac{g_3\sqrt{\pi e}}{V(M)} = \frac{0.353076}{V(M)}.$$

Also in dimension 3, Miyamoto and Kojima proved that Miyamoto's bound in [12] is optimal and that the lowest volume hyperbolic 3–manifold with totally geodesic boundary has boundary a genus 2 surface and volume 6.452 (see [10]). We can compare this optimal bound to the bound obtained using (8) for F_3.

As in our prior analysis in Theorem 6, we obtain a volume bound by finding the common value of $F_3(x) = 4\pi S_3\left(\frac{1}{2}x\right)$. Solving numerically, we obtain a lower bound of 4.079, which is comparable to Miyamoto's optimal bound. This was also observed in [6, Section 7] but, due to the missing factor in the integral formula for F_n (see the footnote on page 1259), the bound obtained there was given as 2.986.

6 Lower bounds for volume of hyperbolic n–manifolds with totally geodesic boundary

We now consider our bounds in general dimension $n \geq 3$. In even dimensions the generalized Gauss–Bonnet theorem gives

$$\mathrm{Vol}(M) = \tfrac{1}{2}|\chi(M)|V_n \geq \tfrac{1}{2}V_n = \frac{(n+1)\pi^{(n+1)/2}}{\Gamma\left(\frac{1}{2}(n+3)\right)}.$$

For odd dimensions, the best lower bound is by Adeboye and Wei [1], with

$$\mathrm{Vol}(M) \gtrsim \left(\frac{2}{n}\right)^{n^2/2}. \tag{9}$$

Miyamoto [12] showed that, for a hyperbolic manifold M with nonempty totally geodesic boundary, we have

$$\mathrm{Vol}(M) \geq \rho_n \,\mathrm{Vol}(\partial M)$$

for some constants ρ_n. In [9, Lemma 1.4.3 and Table 1.4.5] (see also [8]), Kellerhals showed that ρ_n are monotonically increasing with $\rho_6 = 0.64652$. Thus, for $n > 6$ odd, we have

$$\mathrm{Vol}(M) \geq \tfrac{1}{2}\rho_n \tfrac{1}{2}V_{n-1} \geq 0.32326 V_{n-1}.$$

When it applies, this bound is much stronger than (9) (applied to the double of M).

The key ingredient of Miyamoto's proof is his notion of the *hypersphere packings*. These packings have similar properties to the sphere packings in constant curvature spaces. In his paper, Miyamoto proved a hypersphere analogue of the well-known Böröczky's sphere packing theorem, which says that any sphere packing of radius r in an n–dimensional space of constant curvature has density at most that of $n+1$ mutually touching balls in the regular n–simplex of edgelength $2r$ spanned by their centers. Following this line of argument, the constant ρ_n in Miyamoto's volume bound is given by the ratio of the volumes of certain truncated and regular hyperbolic simplices. These volumes can be further related to the volumes of orthoschemes. In her thesis [9], Kellerhals was able to explicitly estimate the latter volumes.

We now show that our results give a new proof of a linear bound for $\mathrm{Vol}(M)$. By Theorem 6, either

$$\mathrm{Vol}(M) \geq \tfrac{1}{4}\log\tfrac{5}{2}\,\mathrm{Vol}(\partial M)$$

or

$$\mathrm{Vol}(M) \geq \tfrac{1}{3} h_n \sqrt{\frac{2\pi e}{n-1}} (\mathrm{Vol}(\partial M))^{(n-2)/(n-1)},$$

where h_n monotonically increases to 1. The first bound is linear and implies, for n odd,

$$\mathrm{Vol}(M) \geq \tfrac{1}{8} \log \tfrac{5}{2}\, V_{n-1}.$$

To show that the second bound also gives us a linear lower bound in terms of V_{n-1}, we note that, by Stirling's approximation,

$$V_n = \frac{(n+1)\pi^{(n+1)/2}}{\Gamma\big(\frac{1}{2}n+3\big)} \leq \frac{1}{\sqrt{2}}\Big(\frac{2\pi e}{n+1}\Big)^{n/2} \leq \frac{1}{\sqrt{2}}\Big(\frac{2\pi e}{n}\Big)^{n/2}.$$

Therefore,

$$\mathrm{Vol}(M) \geq \tfrac{1}{3} h_n \sqrt{\frac{2\pi e}{n-1}} \big(\tfrac{1}{2} V_{n-1}\big)^{(n-2)/(n-1)} \geq \tfrac{1}{3} h_n \cdot \tfrac{1}{2} V_{n-1} = \tfrac{1}{6} h_n V_{n-1}.$$

Thus, for n odd, we have

$$\mathrm{Vol}(M) \geq \min\big(\tfrac{1}{8} \log \tfrac{5}{2}, \tfrac{1}{6} h_n\big) V_{n-1},$$

proving Theorem 8.

This way we obtain another proof of a lower bound linear in V_{n-1} using different methods. The answers are remarkably similar in spite of the different approaches. To compare, our method gives a linear constant tending to $\frac{1}{8} \log \frac{5}{2} \simeq 0.11453$ and Miyamoto and Kellerhals give a slightly better bound of 0.32326. It would be interesting to see if there is any deeper relation between the two.

In conclusion, let us remark that it is widely believed that these bounds for volumes of hyperbolic manifolds, as well as the Gauss–Bonnet bound in even dimensions, are far from sharp. The sharp bounds are known for *arithmetic* orbifolds, and they imply good bounds for arithmetic manifolds (see [3; 4]). These bounds grow superexponentially fast with the dimension. It is not known if there exists a hyperbolic n–manifold whose volume is less than the minimal volume of an arithmetic n–manifold.

References

[1] **I Adeboye**, **G Wei**, *On volumes of hyperbolic orbifolds*, Algebr. Geom. Topol. 12 (2012) 215–233 MR Zbl

[2] **A Basmajian**, *The orthogonal spectrum of a hyperbolic manifold*, Amer. J. Math. 115 (1993) 1139–1159 MR Zbl

[3] **M Belolipetsky**, *Hyperbolic orbifolds of small volume*, from "Proceedings of the International Congress of Mathematicians, II" (S Y Jang, Y R Kim, D-W Lee, I Yie, editors), Kyung Moon Sa, Seoul (2014) 837–851 MR Zbl

[4] **M Belolipetsky**, **V Emery**, *Hyperbolic manifolds of small volume*, Doc. Math. 19 (2014) 801–814 MR Zbl

[5] **M V Belolipetsky**, **S A Thomson**, *Systoles of hyperbolic manifolds*, Algebr. Geom. Topol. 11 (2011) 1455–1469 MR Zbl

[6] **M Bridgeman**, **J Kahn**, *Hyperbolic volume of manifolds with geodesic boundary and orthospectra* 20 (2010) 1210–1230 MR Zbl

[7] **D Calegari**, *Chimneys, leopard spots and the identities of Basmajian and Bridgeman*, Algebr. Geom. Topol. 10 (2010) 1857–1863 MR Zbl

[8] **R Kellerhals**, *Regular simplices and lower volume bounds for hyperbolic n–manifolds*, Ann. Global Anal. Geom. 13 (1995) 377–392 MR Zbl

[9] **R Kellerhals**, *Volumina von hyperbolischen Raumformen*, Habilitationsschrift, Universität Bonn (1995)

[10] **S Kojima**, **Y Miyamoto**, *The smallest hyperbolic* 3*–manifolds with totally geodesic boundary*, J. Differential Geom. 34 (1991) 175–192 MR Zbl

[11] **H Masai**, **G McShane**, *Equidecomposability, volume formulae and orthospectra*, Algebr. Geom. Topol. 13 (2013) 3135–3152 MR Zbl

[12] **Y Miyamoto**, *Volumes of hyperbolic manifolds with geodesic boundary*, Topology 33 (1994) 613–629 MR Zbl

[13] **N G Vlamis**, **A Yarmola**, *The Bridgeman–Kahn identity for hyperbolic manifolds with cusped boundary*, Geom. Dedicata 194 (2018) 81–97 MR Zbl

IMPA
Estrada Dona Castorina, 110, 22460-320 Rio de Janeiro, Brazil

Department of Mathematics, Boston College
Chestnut Hill, MA, 02467, United States

mbel@impa.br, bridgem@bc.edu

Received: 6 July 2020 Revised: 7 January 2021